Statistically Accurate Invariant Measure Informed Forecasting of QG Dynamics

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Abstract: We develop a deep learning surrogate for the Quasi-Geostrophic (QG) system governing oceanic flows. A Fourier Neural Operator (FNO), is trained to learn the system's time derivative for efficient forecasting. We introduce a hybrid loss function that combines a standard Root Mean Squared Error (RMSE) with a Sliced Wasserstein (SW) loss. This SW term improves dynamic consistency by minimizing the statistical divergence between the ground truth and predicted system attractors, yielding a surrogate that captures both short-term accuracy and long-term climate statistics.

QG System

Quasi Geostrategic System[1]:

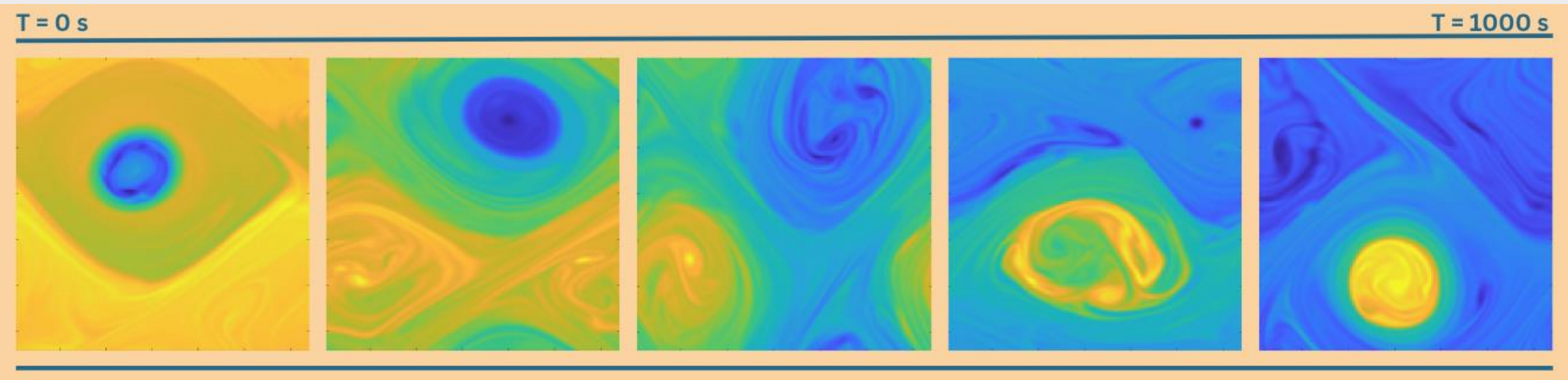
- Study of midlatitude, large-scale **Oceanic Flow**.
- The system formulation is a single primary equation called the quasi-geostrophic barotropic equation. This equation describes the time evolution of **Potential Vorticity**.

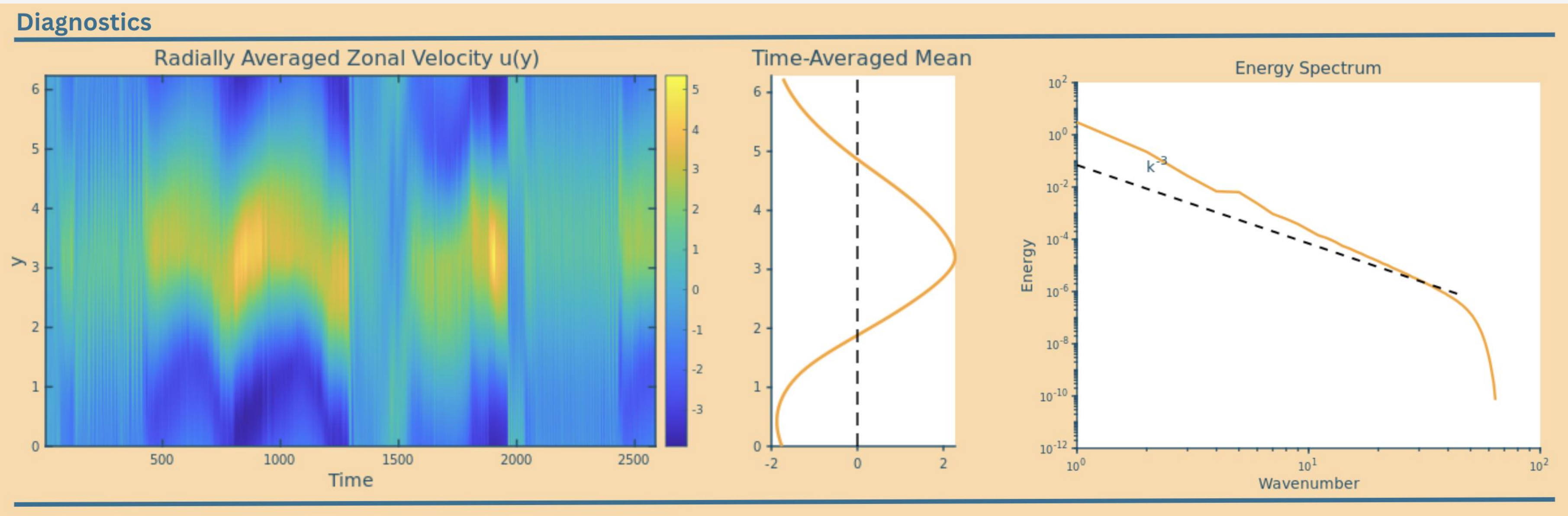
$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{t}} + \boldsymbol{J}(\boldsymbol{\psi}, \boldsymbol{q}) + \beta \frac{\partial \psi}{\partial x} = -d\boldsymbol{\Delta\psi} + f(x,t)$$

QG Visualization & Dataset

Example QG system visualization with the following features:

Resolution	Wind Force	Wind Width	Length of The Domain
128*128	0.40 Nm^{-2}	0.12	2400 km





Training Dataset: Our training dataset contains **256** unique systems. It was generated by applying Latin Hypercube Sampling (**LHS**) to parametrically vary the **Wind Force** and **Wind Width** parameters.

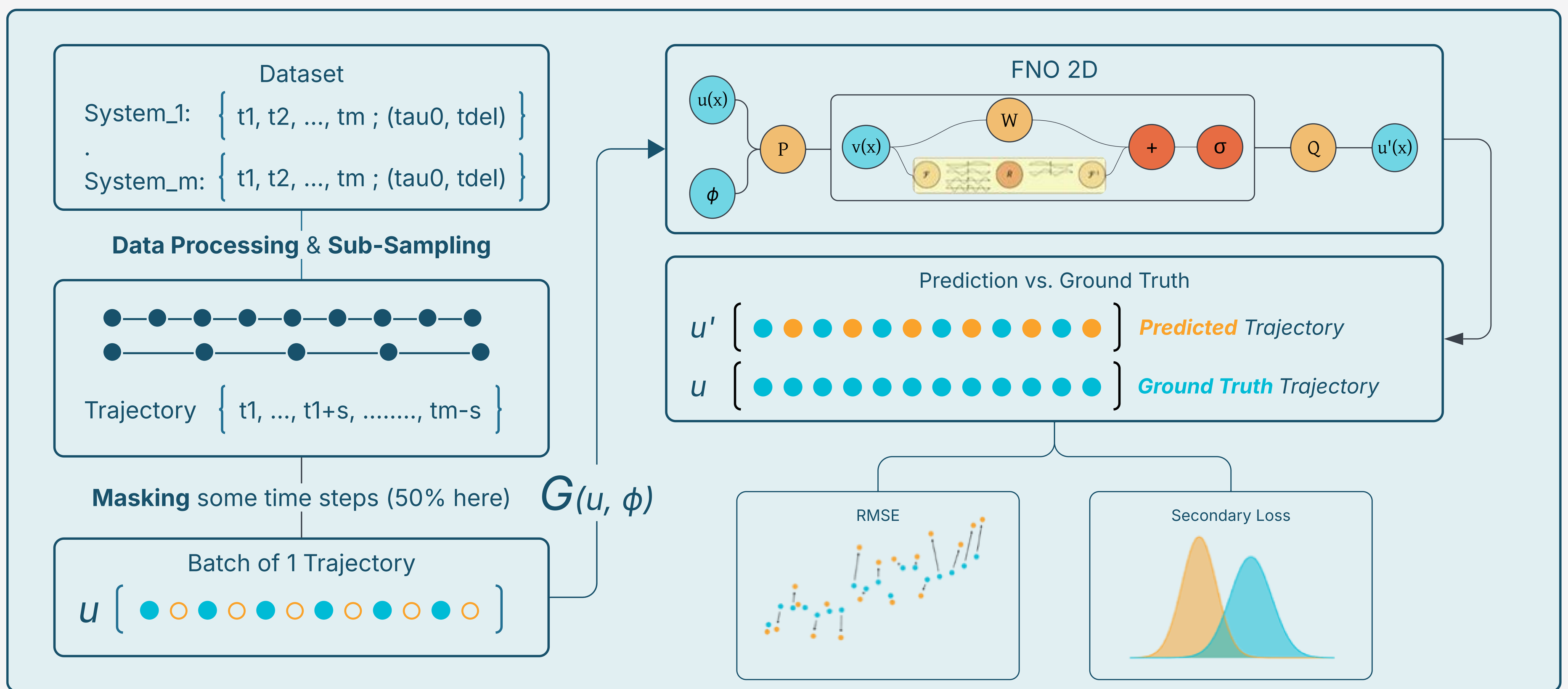
Resolution	Wind Force Range (τ_0)	Wind Width Range (δ)	Dataset Size
64*64	0.02 – 0.40 Nm^{-2}	0.12 – 0.60	256

The Wind Stress formulation: $\boldsymbol{\tau(y) = \tau_0 sech^2(y/\delta)}$

Network Architecture

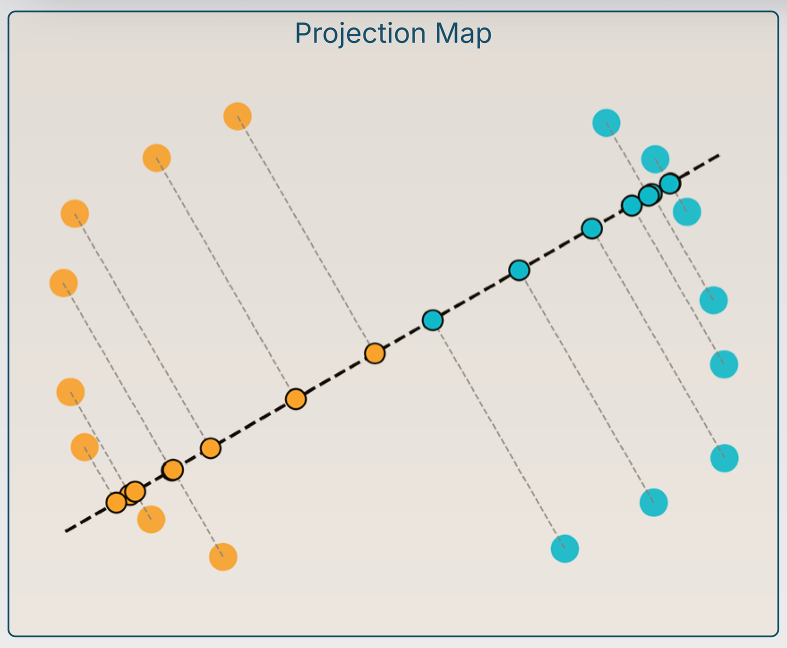
The neural network architecture is primary designed to approximate a surrogate which forecast the next time step of the system via **Interpolation** and learning masked time steps. The network contains 2 main parts:

- Fourier Neural Operator[1] (**FNO**) is used to rebuild a **Masked** trajectory
- **RMSE** Loss + **Secondary** Loss (loss between 2 probability distributions of attractor's invariant measures)



Sliced Wasserstein

$$\widetilde{W}(X,Y) = \int_{\theta \in \Omega} W(X_\theta, Y_\theta)^2 \, d\theta \quad Where \quad X_\theta = \{\langle X_i | \theta \rangle\}_{i \in I} \subset \mathbb{R}$$



The **Sliced Wasserstein**[2] Distance provides a computationally efficient metric for comparing invariant measures associated with complex, high-dimensional attractors. We find the distance by projecting points onto many lines and, for each line, summing the distances between the sorted, corresponding points, then averaging the results.

QG Invariant Measures

Invariant Measures considering for Dynamics-Informed Loss?

We enforce physical consistency by matching the statistical distributions of key dynamical quantities between the ground truth simulation and our surrogate model

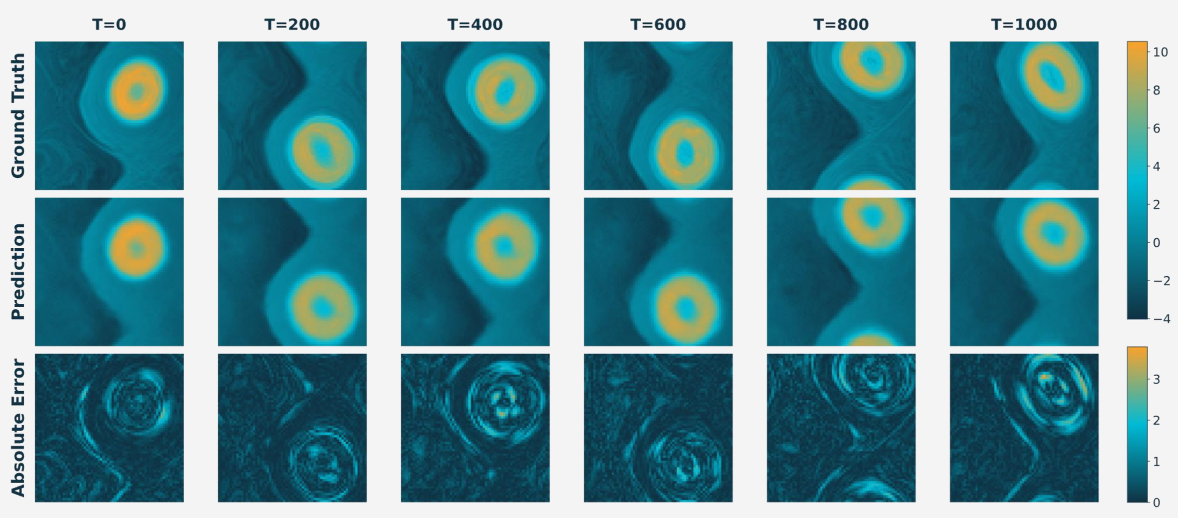
1. **Potential Vorticity** $\mathbf{q}$: The fundamental currency of the flow, defining the eddies and jets.
2. **Nonlinear Advection** $\mathbf{J(\psi, q)}$: The rate of vorticity transport by the velocity field, which governs the formation of complex structures.
3. **Vorticity Tendency** $\frac{\partial \mathbf{q}}{\partial \mathbf{t}}$: The net rate of change of the system, encapsulating the combined effects of advection, dissipation, and forcing.

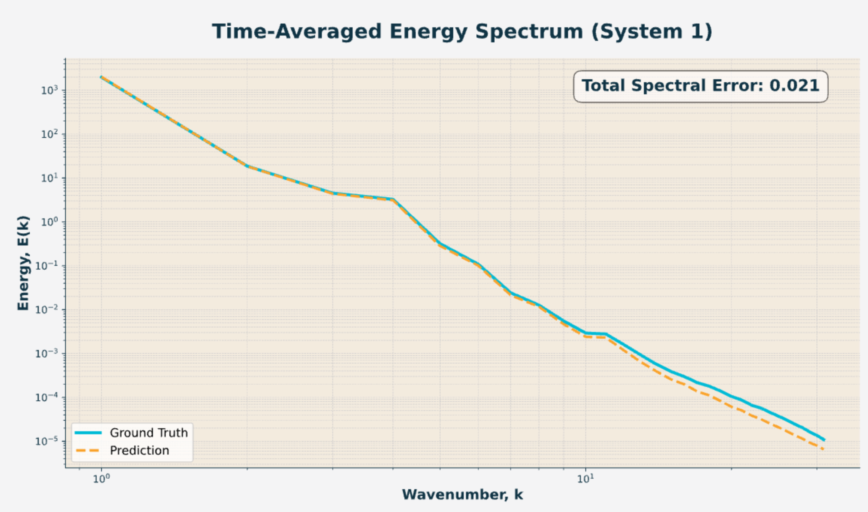
Final Loss: $\boldsymbol{L = RMSE + \lambda \sum_i \widetilde{W}\,(u_i, u'_i)}$

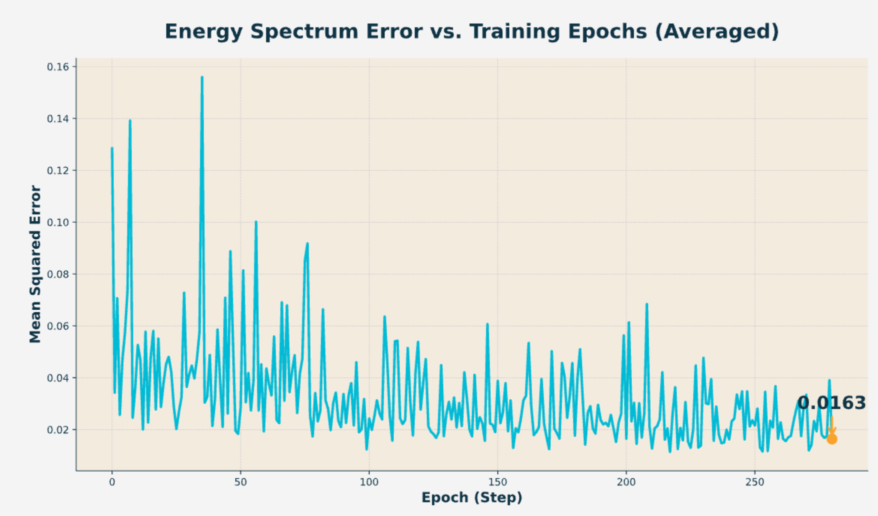
Results

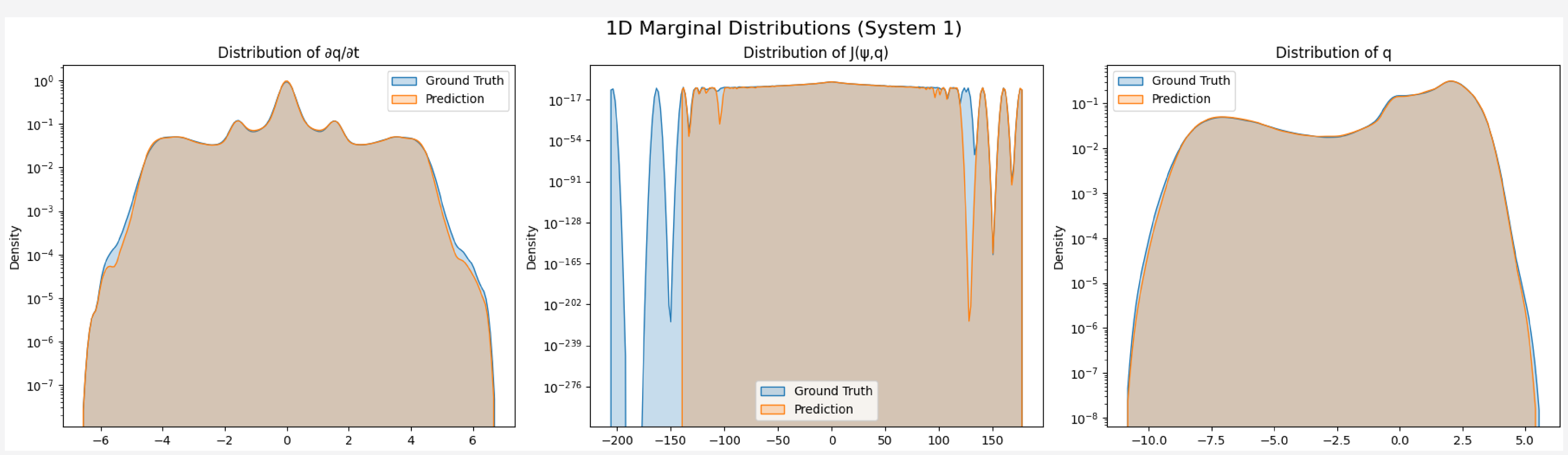
Experiment Setup:

Resolution	64*64
Time Steps	500T
Sub-sampling	20
dt	2s
Time Horizon	0-1000s
Training Systems	200
Testing Systems	25









Ongoing Research

Our ongoing research focuses on three key areas. First, we are analyzing the effect of data **subsampling** on model performance. Second, we are developing models for **extrapolative forecasting**, using our dynamics-informed loss to enhance long-term stability. Finally, train on **analytical derivation** of the invariant measure of the QG through exact conserved quantities and statistical mechanics (subtract mean and derive probability distribution for the fluctuations)

Acknowledgement

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2. Purdue University, West Lafayette, IN, United States

* This work is a follow-up to the “Training neural operators to preserve invariant measures of chaotic attractors”[3]

References

[1] Li, Z., et al. "Fourier neural operator for parametric partial differential equations," in arXiv preprint arXiv:2010.08895, 2020.
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[3] Jiang, R., et al. "Training neural operators to preserve invariant measures of chaotic attractors," in Advances in Neural Information Processing Systems, vol. 36, pp. 27645–27669, 2023.